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Some remark on locally o-minimal structures

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概要

abstract Locally o-minimal structures are some local versions of o-minimal structures. This notion is studied, e.g. in [1], [2]. O-minimal structures are characterized by some independence notions. We consider whether locally o-minimal structures are characterized in the same way.

1. Introduction and preliminaries

Locally o-minimal structures are some local versions of (weakly) o-minimal structures. We recall some definitions at first.

Definition 1 A linearly ordered structure $M = (M, <, \cdots)$ is $o-minimal$ if every definable subset of M is the union of finitely many points and open intervals.

A linearly ordered structure $M = (M, <, \cdots)$ is $weakly\ o - minimal$ if every definable subset of M is the union of finitely many convex sets.

Definition 2 M is $locally\ o - minimal$ if for any definable set $A \subset M$ and $a \in M$, there is an open interval $I \ni a$ such that $I \cap A$ is a finite union of intervals and points.

M is $strongly\ locally\ o - minimal$ if for any $a \in M$, there is an open interval $I \ni a$ such that whenever A is a definable subset of M, then $I \cap A$ is a finite union of intervals and points.

M is called $uniformly\ locally\ o - minimal$ if for any $\varphi(x, \bar{y}) \in L$ and any $a \in M$, there is an open interval $I \ni a$ such that $I \cap \varphi(M, \bar{b})$ is a finite union of intervals and points for any $\bar{b} \in M^n$.

Example 3 The following examples are shown in [1] and [2].

$(\mathbf{R}, +, <, \mathbf{Z})$ and $(\mathbf{R}, +, <, \sin)$ are locally o-minimal structures.

Let $L = \{<\} \cup \{P_i : i \in \omega\}$ where P_i is a unary predicate. Let $M = (\mathbf{Q}, <^M, P_0^M, P_1^M, \dots)$ be the structure defined by $P_i^M = \{a \in M : a < 2^{-i}\sqrt{2}\}$. Then M is uniformly locally o-minimal, but it is not strongly locally o-minimal.

It is proved that (weakly) o-minimal structures have no independence property. And there are some characterizations of o-minimal structures by independence relation (geometric property).

Problem 4 Can we characterize locally o-minimal structures by independence relation ?

2. Forking in o-minimal structures

There is a result about forking in o-minimal structures in [9].

We recall some definitions.

Definition 5 A formula $\varphi(\bar{x}, \bar{a})$ *divides* over a set A if there is a sequence $\{\bar{a}_i : i \in \omega\}$ with $tp(\bar{a}_i/A) = tp(\bar{a}/A)$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is k-inconsistent for some $k \in \omega$.

A formula $\phi(\bar{x}, \bar{a})$ $forks$ over A if $\phi(\bar{x}, \bar{a}) \vdash \bigvee_{i<n} \psi_i(\bar{x}, \bar{b}_i)$ and each $\psi_i(\bar{x}, \bar{b}_i)$ divides over A.

Definition 6 An o-minimal theory has $densely\ ordered\ definable\ closures$ if for all $A \subset \mathcal{M}$, the ordering induced from $\mathcal{M}$ on $dcl(A)$ is dense and without endpoints, where $\mathcal{M}$ is the big model and dcl is the definable closure.

Any o-minimal theory with group structure has this property.

Definition 7 An o-minimal theory is *nice* if it has densely ordered definable closures and if for all sets $A \subset \mathcal{M}$ and any two noncuts $p(x), q(x) \in S_1(A)$, there is an A-definable function f such that a realizes $p(x)$ then $f(a)$ realizes $q(x)$.

Any o-minimal theory with field structure is nice.

The set $x < a$ is send to $-a > x$ by $f(x) = -x$. And the interval $x > dcl(A)$ is send to bounded $x > 0$ and $x < dcl(A) \cap (0, \infty)$ by $f(x) = 1/x$, and $x < a$ is send to $x < b$ by $f(x) = x - a + b$.

Definition 8 Let $\phi(\bar{x}, \bar{a})$ define a cell in an o-minimal structure. We define $\phi(\bar{x}, \bar{a})$ is $halfway-definable$ over a set B inductively on $n = l(\bar{x})$;

If $n = 1$, then $\phi(x, \bar{a})$ is an interval I. I is halfway-definable over B if one of its endpoints is in $dcl(B) \cup \{\infty, -\infty\}$.

If $n = m+1$, then $\phi(\bar{x}, \bar{a})$ is given by the graph of a function $f(\bar{y}, \bar{a})$ on a cell $X \subset \mathcal{M}^m$ or by the region of two functions $f_1(\bar{y}, \bar{a}) < f_2(\bar{y}, \bar{a})$ on X. We say that $\phi(\bar{x}, \bar{a})$ is $halfway-definable$ over a set B, in the first case, if $f(\bar{y}, \bar{a})$ is B-definable, in the second case, if one of the $f_i(\bar{y}, \bar{a})$ is B-definable.

Definition 9 A formula $\phi(\bar{x}, \bar{a})$ is *good* over a set B if there is a cell $X \subset \mathcal{M}^n$ which is halfway-definable over B such that $X \subset \phi(\mathcal{M}^n, \bar{a})$.

Theorem 10 [9]

Let T be a nice o-minimal theory.

For a formula $\phi(\bar{x}, \bar{a})$ and a set B, the following are equivalent ;

1. $\phi(\bar{x}, \bar{a})$ forks over B.

2. $\phi(\bar{x}, \bar{a})$ is not good over B.

3. There are $\bar{a}_0, \ldots, \bar{a}_{m-1}$ such that for any $i < m$, $tp(\bar{a}_i/B) = tp(\bar{a}/B)$ and $\bigwedge_{i<m} \phi(\bar{x}, \bar{a}_i)$ is inconsistent.

But in o-minimal structures, this nonforking satisfies neither the symmetry nor the transitivity in general.

This argument of forking is concrete and depends on properties of o-minimal structures, e.g. the monotonicity theorem and the cell decomposition theorem. Thus if we try the analogous argument, we need to modify these theorems to local context. There are some modifications of them in [2] and [1] for strongly locally o-minimal structures.

3. þ-forking in o-minimal structures

There is another kind of forking, thorn-forking. It is known that this forking notion is available to NIP theories, or theories with the strict order property.

Definition 11 A formula $\phi(\bar{x}, \bar{a})$ *strongly divides* over A if $tp(\bar{a}/A)$ is nonalgebraic and $\{\phi(x, \bar{a}')\}$ with $tp(\bar{a}/A) = tp(\bar{a}'/A)$ is k-inconsistent for some $k < \omega$.

A formula $\phi(\bar{x}, \bar{a})$ þ-divides (thorn divides) over A if for some tuple $\bar{c}$, $\phi(\bar{x}, \bar{a})$ strongly divides over $A\bar{c}$.

A formula $\phi(\bar{x}, \bar{a})$ þ-forks over A if it implies a finite disjunction of formulas which þ-divides over A.

Definition 12 For a formula ϕ, a set Δ of formulas in variables $\bar{x}$, $\bar{y}$, a set of formulas Π in the variables $\bar{y}$, $\bar{z}$ ($\bar{z}$ may be infinite) and a number $k < \omega$, we define þ(ϕ, Δ, Π, k) inductively as follows :

(1) þ$(\phi, \Delta, \Pi, k) \geq 0$ if ϕ is consistent.

(2) For any ordinal α, þ$(\phi, \Delta, \Pi, k) \geq \alpha + 1$ if there is a $\delta \in \Delta$, some $\pi(\bar{y}, \bar{z}) \in \Pi$ and parameters $\bar{c}$ such that

(*a*) þ$(\phi \wedge \delta(\bar{x}, \bar{a}), \Delta, \Pi, k) \geq \alpha$ for infinitely many $\bar{a} \models \pi(\bar{y}, \bar{c})$.

(*b*) $\{\delta(\bar{x}, \bar{a})\}_{\bar{a} \models \pi(\bar{y}, \bar{c})}$ is $k-$inconsistent.

(3) For any λ limit ordinal, þ$(\phi, \Delta, \Pi, k) \geq \lambda$ if þ$(\phi, \Delta, \Pi, k) \geq \beta$ for all $\beta < \lambda$.

(4) $\text{þ}(\phi, \Delta, \Pi, k) = \infty$ if it is bigger than all the ordinals.

As usual, for a type p, we define $\text{þ}(p, \Delta, \Pi, k) = \min\{ \text{þ}(\phi, \Delta, \Pi, k) \mid \phi(\bar{x}) \in p\}$.

Definition 13 Let T be a complete theory of some language L. If $\text{þ}(\phi, \Delta, \Pi, k)$ is finite for any type $p(\bar{x})$, any finite sets Δ and Π and any finite k, then we call such a theory *rosy*.

Theorem 14 [10]

Let T be rosy. And let p be a type over $B \supset A$.

Then p does not þ-fork over A if and only if $\text{þ}(p \restriction A, \Delta, \Pi, k) = \text{þ}(p, \Delta, \Pi, k)$ for all finite sets Δ, Π of formulas and for all k.

Theorem 15 [10]

þ-independence defines an independence relation in any rosy theory. That is, þ-forking satisfies such axioms : Existence, Extension, Reflexivity, Monotonicity, Finite character, Symmetry, Transitivity.

Definition 16 We define $U^{\text{þ}}$-rank (U thorn rank) inductively as follows.

Let $p(\bar{x})$ be a type over A. Then

(1) $U^{\text{þ}}(p(\bar{x})) \geq 0$ if $p(\bar{x})$ is consistent.

(2) For any ordinal α, $U^{\text{þ}}(p(\bar{x})) \geq \alpha + 1$ if there is some tuple $\bar{a}$ and some type $q(\bar{x}, \bar{a})$ over $A\bar{a}$ such that $q(\bar{x}, \bar{a}) \supset p(\bar{x})$, $U^{\text{þ}}(q(\bar{x}, \bar{a})) \geq \alpha$, and $q(\bar{x}, \bar{a})$ þ-forks over A.

(3) For any λ limit ordinal, $U^{\text{þ}}(p(\bar{x})) \geq \lambda$ if $U^{\text{þ}}(p(\bar{x})) \geq \beta$ for all $\beta < \lambda$.

Definition 17 A theory T is *superrosy* if $U^{\text{þ}}(p(\bar{x})) < \infty$ for any type $p(\bar{x})$.

I introduce a result that has the relation with o-minimal structures.

Theorem 18 [10]

Let M be an o-minimal structure.

For any definable $A \subset M^n$, $U^{\text{þ}}(A) = dim(A)$ in the sense of o-minimal structure.

Sketch of proof ;

Let $A \subset M^m$ be a definable set with $dim(A) = n$. We show $U^{\text{þ}}(A) \leq n$. Suppose not. Let $i < \omega$ be minimal such that for some definable set A with $dim(A) = i$ and $U^{\text{þ}}(A) \geq i+1$, and A is defined by $\phi(\bar{x})$ over B. So by minimality of i, for some generic point $\bar{a} \in A$, $U^{\text{þ}}(\text{tp}(\bar{a}/B)) \geq i + 1$ and $dim(\text{tp}(\bar{a}/B)) \geq i$. By some suitable projection from M^m to M^i, we can assume that $A \subset M^i$ and A is open. As $U^{\text{þ}}(\text{tp}(\bar{a}/B)) \geq i+1$, there are a formula $\delta(\bar{x}, \bar{y})$, and a tuple $\bar{b}_0$ and some $C \supset B$ such that ; $\text{tp}(\bar{b}_0/C)$ is nonalgebraic, $U^{\text{þ}}(\text{tp}(\bar{a}/B) \cup \{\delta(\bar{x}.\bar{b}_0)\}) \geq i$ and $\{\delta(\bar{x}, \bar{b}')\}_{\bar{b}' \models \text{tp}(\bar{b}_0/C)}$ is $k-$inconsistent for some $k < \omega$.

Case.1 $i = 1$

Take an indiscernible sequence $I = \{\bar{b}_j : j < \omega\}$ with $\bar{b}_0 \equiv \bar{b}_j\,(C)$, and let A_j be the *definable set by $\delta(x, \bar{b}_j)$. By o-minimality, A_j contains an interval. We can take a set $X \subset M$ such that X is a set consisting of the left endpoints of first intervals contained in A_js. By $k-$inconsistency, X is infinite but contains no interval. Contradiction.*

Case.2 general i

Let π_j be the projection of M^i to the jth coordinate. We consider the formula $\phi_0(x_0, \bar{b}')$ with $\bar{b}' \equiv \bar{b}_0 (C)$ such that $\phi_0(x_0, \bar{b}') := \exists \bar{x}\{\phi(\bar{x}) \wedge \delta(\bar{x}, \bar{b}') \wedge \pi_0(\bar{x}) = x_0\}$. This formula defines an inifinite set, and by Case.1, $\{\phi_0(x_0, \bar{b}')\}_{\bar{b}' \models \mathrm{tp}(\bar{b}_0/C)}$ is not $k-$inconsistent for any $k < \omega$. Thus we can find an infinite subsequence $J = \{\bar{b}_j\}_{j \in J}$ of I such that $\{\phi_0(x_0, \bar{b}_j)\}_{j \in J}$ is consistent. Let it be realized by a_0. And let $q_0(\bar{y})$ be the type $\mathrm{tp}(\bar{b}_0/C) \cup \{\phi_0(a_0, \bar{y})\}$ and consider $\{\delta(\bar{x}, \bar{b}')\}_{\bar{b}' \models q_0}$. Next we define the formula $\phi_1(x_1, \bar{b}') := \exists \bar{x}\{\phi(\bar{x}) \wedge \delta(\bar{x}, \bar{b}') \wedge \pi_1(\bar{x}) = x_1 \wedge \pi_0(\bar{x}) = a_0\}$. So $\{\phi_1(x_1, \bar{b}')\}_{\bar{b}' \models q_0}$ is not $k-$inconsistent for any $k < \omega$. Let it be realized by a_1 and let $q_1(\bar{y}) := q_0(\bar{y}) \cup \{\phi_1(a_1, \bar{y})\}$. We can iterate this procedure. Thus there are nonalgebraic types $q_0 \subset q_1 \subset \cdots \subset q_{i-1}$ and a point $(a_0, \cdots, a_{i-1}) \in M^i$ which realizes infinitely many $\delta(\bar{x}, \bar{b}')$s. Contradiction.

For other inequality, let $A \subset M^k$ and $dim(A) = n$. We can find a suitable projection π from M^k to M^n, so we may assume that $\pi(A)$ contains an open $n-$box B. We can prove $U^p(B) \geq n$ inductively. Let π_n be the projection to the last coordinate. For any $a \in \pi_n(B)$, $\pi_n^{-1}(a) \cap B$ is an open $(n-1)-$box. And as $x_n = a$ is a þ-forking formula, by the induction hypothesis, $U^p(\pi_n^{-1}(a) \wedge B) = n - 1$. Then $U^p(B) \geq n$ and by the monotonicity $U^p(A) \geq n$. ∎

There are characterizations of o-minimal structures, or structures having o-minimal open core in relation to rosyness, e.g. in [11].

Many times, for locally o-minimal structures M, we recognize that there is a definable infinite discrete unary set in M to witness non-o-minimality of M. This verification is impossible for M which has the (global) independence relation satisfying the symmetry.

Definition 19 Let M be a structure of some language L. We say that M satisfies the *exchange property* if for any $a, b \in M$ and subset $C \subset M$, if $b \in acl(C \cup \{a\})$ and $b \notin acl(C)$, then $a \in acl(C \cup \{b\})$, where acl is the algebraic closure in the sense of model theory.

The next fact is well known.

Theorem 20 [4]

Let M be an o-minimal structure. Then M satisfies the exchange property.

Fact 21 [3]

Let M be an expansion of a dense linear ordered structure and let $Th(M)$ be the theory of M. Suppose that an infinite discrete unary ordered set is definable in M. Then $Th(M)$ can *not satisfy the exchange property.*

Proof ;

Suppose that an infinite discrete unary set A is defined using finite parameters S in a sufficiently large saturated model M of $Th(M)$. By saturation, there is $a \in A \setminus acl(S)$. As A is discrete, there is an interval I such that $I \cap A = \{a\}$. And by saturation again, there is $b \in I \setminus acl(S \cup \{a\})$ such that $a < b$. Thus as $a = max(A \cap (-\infty, b))$, $a \in acl(S \cup \{b\}) \setminus acl(S)$. But $b \notin acl(S \cup \{a\})$. ∎

4. Further problems

There is a characterization of groups definale in o-minimal structures by using forking in NIP theories in [13] and [14]. But they replace forking of complete types by that of measures.

Definition 22 Let $\mathcal{M}$ be a sufficiently large saturated model. And let X be a sort or $\emptyset-$definable set in $\mathcal{M}$.

$Def(X)$ denote the subsets of X definable in $\mathcal{M}$, and $Def_A(X)$ those sets defined over $A \subset \mathcal{M}$.

A *Keisler measure* μ on X over A is a finitely additive probability measure on $Def_A(X)$, that is, a map μ from $Def_A(X)$ to the interval $[0,1]$ such that $\mu(\emptyset) = 0$, $\mu(X) = 1$ and for Y, $Z \in Def_A(X)$, $\mu(Y \cup Z) = \mu(Y) + \mu(Z) - \mu(Y \cap Z)$.

A *global Keisler measure* on X is a finitely additive probability measure on $Def(X)$.

Definition 23 Let $\mathcal{M}$ be as above. And let μ be a Keisler measure over B and $A \subset B \subset \mathcal{M}$. We say that μ *does not divide over* A if whenever $\phi(\bar{x}, \bar{b})$ is over B and $\mu(\phi(\bar{x}, \bar{b})) > 0$, then $\phi(\bar{x}, \bar{b})$ does not divide over A. Similary we say that μ *does not fork over* A if every formula of positive $\mu-$measure does not fork over A.

Problem 24 Can we characterize locally o-minimal structures by measure forking ?

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